Problems for Ultrahyperbolic Equations in Half-Space

Prof. Dmitry P. Kostomarov

Faculty of Computational Mathematics and Cybernetics
Lomonosov Moscow State University

Pontryagin Anniversary Conference
June 17–22, 2008, Moscow, Russia
Section “Differential Equations”
Subsection “Partial Differential Equations”

Introduction

- Ultrahyperbolic equations are set aside by mathematicians since they are usually not met in applications. At the same time such equations are very interesting from mathematical point of view, because their solutions possess both hyperbolic and elliptic properties.
- Let us consider ultrahyperbolic equation of dimension 3×2:

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

- In the absence of the dependence on y or z we get three-dimensional equation of oscillations.
- If in addition there is no dependence on variable x_3, then we get two-dimensional equation of oscillations.
- When the dependence on variables y and z or on $\mathbf{x}$ is absent we get the Laplace equation. These are particular, “degenerate” cases.
- I was interested in answering the question: how dualism of ultrahyperbolic equations reveals in general case, when the solution depends on all five variables.
- We consider it in the example of problems in half-space $z \geqslant 0$.
- Academician L.A. Artsimovich used to say “Science is the best modern way to satisfy personal curiosity at government expenses”.

1. Cauchy problem

1.1. Formulation of the Cauchy problem

- The Cauchy problem:

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \tag{1.1}$$

$$u(\mathbf{x}, y, 0) = f(\mathbf{x}, y), \tag{1.2}$$

$$\frac{\partial u}{\partial z}(\mathbf{x}, y, 0) = g(\mathbf{x}, y). \tag{1.3}$$

- Fourier transformation over variable y:

$$\hat{u}(\mathbf{x}, k, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iky} u(\mathbf{x}, y, z)\, dy, \tag{1.4}$$

$$\hat{f}(\mathbf{x}, k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iky} f(\mathbf{x}, y)\, dy, \tag{1.5}$$

$$\hat{g}(\mathbf{x}, k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iky} g(\mathbf{x}, y)\, dy. \tag{1.6}$$

- Reduced problem:

$$\frac{\partial^2 \hat{u}}{\partial z^2} = \frac{\partial^2 \hat{u}}{\partial x_1^2} + \frac{\partial^2 \hat{u}}{\partial x_2^2} + \frac{\partial^2 \hat{u}}{\partial x_3^2} + k^2 \hat{u}, \tag{1.7}$$

$$\hat{u}(\mathbf{x}, k, 0) = \hat{f}(\mathbf{x}, k), \tag{1.8}$$

$$\frac{\partial \hat{u}}{\partial z}(\mathbf{x}, k, 0) = \hat{g}(\mathbf{x}, k). \tag{1.9}$$

- Hyperbolic equation (1.7) can be interpreted as an oscillation equation with the source, linearly dependent on the solution. This term differs it from usual equation of oscillations.

1.2. Solution of the problem (1.7)–(1.9)

- The scheme of solution of the problem (1.7)–(1.9):
  1. In spherically symmetric case in $\mathbf{x}$ space the problem becomes two-dimensional (independent variables r and z). In this case for equation (1.7) the Riemann function is known, allowing writing out the solution.
  2. The solution of the general problem can be obtained from the solution in spherically symmetric case with the averaging method.
  3. The solution in the case of two-dimensional space $\mathbf{x}$ is constructed by the descent method.
- The solution of the problem (1.7)–(1.9) in the three-dimensional case:

$$\hat{u}(\mathbf{x}, k, z) = \hat{u}_1(\mathbf{x}, k, z) + \hat{u}_2(\mathbf{x}, k, z) + \hat{u}_3(\mathbf{x}, k, z), \tag{1.10}$$

$$\hat{u}_1(\mathbf{x}, k, z) = \frac{\partial}{\partial z}\left(\frac{z}{4\pi}\iint_{S_{\mathbf{x}}^z}\hat{f}(\xi, k)\,d\Omega\right), \tag{1.11}$$

$$\hat{u}_2(\mathbf{x}, k, z) = \frac{1}{4\pi}\iint_{S_{\mathbf{x}}^z}\left(\frac{1}{2}k^2z^2\hat{f}(\xi, k) + z\hat{g}(\xi, k)\right)d\Omega, \tag{1.12}$$

$$\hat{u}_3(\mathbf{x}, k, z) = \frac{1}{4\pi}\iiint_{K_{\mathbf{x}}^z}\left(k^4 z\frac{I_2(t)}{t^2}\hat{f}(\xi, k) + k^2\frac{I_1(t)}{t}\hat{g}(\xi, k)\right)d\xi, \tag{1.13}$$

$$t = k\sqrt{z^2 - s^2}. \tag{1.14}$$

- Here $S_{\mathbf{x}}^r$ and $K_{\mathbf{x}}^r$ are the sphere and the ball of radius r with the centre in point $\mathbf{x}$, $s = s(\mathbf{x}, \xi)$ is the distance between point $\mathbf{x}$ — the centre of the sphere or the ball — and current point of integration ξ.
- Functions I_1 and I_2 are Bessel functions of imaginary argument, entering the only summand $\hat{u}_3$.

1.3. Analyses of the constructed solution

1. Obtained formulas show that the problem (1.7)–(1.9) has classic or generalized solution in the same functional spaces as the Cauchy problem for oscillation equation.
2. Derivation of formulas (1.10)–(1.13) with the Riemann and averaging methods proves the uniqueness of the solution.
3. Separate summands in sum (1.10) do not satisfy equation (1.7). For example, the first term in sum (1.10) satisfies oscillations equation, but not (1.7).
4. Poisson formula, which gives solution of the Cauchy problem for oscillation equation, contains only surface integrals of type (1.11) and (1.12). Existence in the solution of the problem (1.7)–(1.9) of volume integral over the ball (1.13) together with surface integrals leads to existence of the substantial difference in the solution properties. This is seen particularly well in the problems with local initial conditions.

1.4. The problem with local initial conditions

Let initial functions $\hat{f}(\mathbf{x}, k)$ and $\hat{g}(\mathbf{x}, k)$ be non-zero in some bounded area D and let $R_1(\mathbf{x})$ and $R_2(\mathbf{x})$ be the smallest and the largest distances from point $\mathbf{x}$ to points of area D. In this case the solution of our problem goes through three stages:

1. *Stage 1.* $0 \leqslant z < R_1(\mathbf{x})$. For such z the solution at point $\mathbf{x}$ equals zero. The effect of the finite speed of spreading is typical for hyperbolic equations.
2. *Stage 2.* $R_1(\mathbf{x}) < z < R_2(\mathbf{x})$. In this stage the solution is determined by formulas (1.10)–(1.14). It is qualitatively similar to the solution of the Cauchy problem for usual oscillation equation.
3. *Stage 3.* $R_2(\mathbf{x}) < z < \infty$. For such values of z the sphere $S_{\mathbf{x}}^z$ in formulas (1.11)–(1.12) does not have common points with area D, and ball $K_{\mathbf{x}}^z$ contains area D on a whole. As the result summands $\hat{u}_1$ and $\hat{u}_2$ in sum (1.10) turn into zero and integral over the ball $K_{\mathbf{x}}^z$ (1.13) turns into integral over the area D. The solution of the problem (1.7)–(1.9) in this stage takes a simple form:

$$\hat{u}(\mathbf{x}, k, z) = \frac{1}{4\pi} \iiint_D \left(k^4 z \frac{I_2(t)}{t^2} \hat{f}(\xi, k) + k^2 \frac{I_1(t)}{t} \hat{g}(\xi, k) \right) d\xi, \qquad (1.15)$$

   where $t = k\sqrt{z^2 - s^2}$, $s = s(\mathbf{x}, \xi)$.

Formula (1.15) does not contain integrals over time dependent areas: sphere $S_{\mathbf{x}}^z$ and ball $K_{\mathbf{x}}^z$. Only parametric dependence of the solution on $\mathbf{x}$ and z is left through the variable t. As the result the solution looses wave nature and elliptic properties start to develop in it.

1.5. Asymptotic formula for integral (1.15)

- Let us consider area T bounded in space X with no common points with the area D.
- Let us denote R_1 and R_2 — the lowest and the largest distance between points of areas D and T closure, d — diameter of area D and suppose validity of inequalities:

$$d \ll R_1, \quad d \ll z - R_2. \tag{1.16}$$

- The first inequality means that the distance between the areas is much greater than the size of the area D and the second — that a considerable time is passed from the beginning of the third stage at all points of the area T.
- For the solution in the third stage (1.15) at large z (1.16) asymptotic formula is valid:

$$\hat{u}(\mathbf{x}, k, z) \approx C_1(k)k^3 w_1(r, k, z) + C_2(k)k^2 w_2(r, k, z). \tag{1.17}$$

Here

$$w_1(r, k, z) = \frac{kzI_2(k\sqrt{z^2 - r^2})}{k^2(z^2 - r^2)}, \tag{1.18}$$

$$w_2(r, k, z) = \frac{I_1(k\sqrt{z^2 - r^2})}{k\sqrt{z^2 - r^2}}, \tag{1.19}$$

$$C_1(k) = \frac{1}{4\pi}\iiint_D \hat{f}(\xi, k)\, d\xi, \quad C_2(k) = \frac{1}{4\pi}\iiint_D \hat{g}(\xi, k)\, d\xi. \tag{1.20}$$

- Formulas (1.17)–(1.20) show that the solution simplifies away from the area D with z growth: it becomes spherically symmetric with standard dependence on variables r and z. At that the initial functions enter the result only as integral coefficients C_1 and C_2 (1.20).
- It is easily to make sure that functions $w_1(r, k, z)$ and $w_2(r, k, z)$ satisfy equation (1.7) in spherically symmetric case.

1.6. Solution of the original Cauchy problem (1.1)–(1.3)

- The solution of the problem (1.1)–(1.3) is found with the inverse Fourier transformation:
$$u(\mathbf{x}, y, z) = \int_{-\infty}^{+\infty} e^{iky}\, \hat{u}(\mathbf{x}, k, z)\, dk. \tag{1.22}$$
- The problem of existence of the solution reduces to the problem of convergence of this integral.
- Let K_1 be one-dimensional space: $-\infty < k < +\infty$, $O_4 = X_3 \times K_1$. Assume that initial functions (1.2), (1.3) are decomposable in Fourier integral over variable y and make the following assumptions about their Fourier image:
$$\hat{f}(\mathbf{x}, k) \in C^1(X_3) \cap C(O_4), \quad \hat{g}(\mathbf{x}, k) \in C(O_4), \tag{1.23}$$
$$|\hat{f}(\mathbf{x}, k)| \leqslant f_0 e^{-|k|Z}, \quad |\hat{g}(\mathbf{x}, k)| \leqslant g_0 e^{-|k|Z}, \tag{1.24}$$
$$|\operatorname{grad} \hat{f}(\mathbf{x}, k)| \leqslant h_0 L(k). \tag{1.25}$$
- Here f_0, g_0, h_0 are constants, $L(k)$ is some function, integrable over variable k with infinite limits. In this case the solution of the problem (1.7)–(1.9) is a continuous function, satisfying at any $\mathbf{x} \in X_3$ to inequality:
$$|\hat{u}(\mathbf{x}, k, z)| \leqslant \left(f_0 \cosh kz + \frac{g_0}{k} \sinh kz \right) e^{-|k|Z} + h_0 z L(k). \tag{1.26}$$
- This inequality provides convergence of integral (1.22) at half-interval $0 \leqslant z < Z$. The solution is constructed.
- Uniqueness follows from the previous results. Continuous dependence on initial conditions — from inequality (1.26).

1.7. The problem with local initial conditions

- Let the initial functions f, g be non-zero only in bounded area $D \subset X_3$ at any y.
- Denote D_1 — the area, consisting of the points which have the distance from area D no more than Z.
- The solution of the problem outside area D_1 equals zero due to the finite speed of perturbations spreading.
- Let us consider in space X_3 bounded area T: $D \subset T \subset D_1$. Denote ρ is the largest distance between points of areas D and T closure.
- In the case, when $z_1 = \rho < Z$, with z change in interval (z_1, Z) the solution of the Cauchy problem at all points of area T comes to the third stage.
- If the difference $(Z - z_1)$ is large enough, then the interval (z_1, Z) can be split in the three sub-areas:

$$z_1 < z < z_2, \quad z_2 \leqslant z \leqslant z_3, \quad z_3 < z < Z. \tag{1.27}$$

- In the first sub-area, adjoining the left boundary of the interval, the solution gradually looses its hyperbolic properties. In the second sub-area it is practically independent on variables **x** and weakly depends on variables y and z. Finally, in the third sub-area near the boundary of solution existence Z its dependence on all arguments again becomes sharper.

1.8. Example

- The described properties of the solution of the Cauchy problem can be illustrated with the example of simple spherically symmetric problem with initial conditions:

$$f(r, y) = 0, \quad g(r, y) = \frac{g_0 l^2}{l^2 + y^2} \eta(r_0 - r), \tag{1.28}$$

where $\eta = \eta(\zeta)$ is the Havyside step function. Function g is non-zero only inside the ball of radius r_0. Its Fourier image over variable y is:

$$\hat{g}(r, k) = \frac{g_0 l}{2} e^{-|k| l} \eta(r_0 - r). \tag{1.29}$$

- The solution of the problem (1.7)–(1.9) in the third stage is given by the formula:

$$\hat{u}(r, k, z) = \frac{g_0 l}{4r} e^{-|k| l} \int_{-r_0}^{+r_0} I_0 \left(k \sqrt{z^2 - (s - r)^2} \right) s\, ds. \tag{1.30}$$

- For construction of the solution of the problem (1.1)–(1.3) let us make inverse Fourier transformation over y:

$$u(r, y, z) = \frac{g_0 l}{4r} \int_{-r_0}^{+r_0} s\, ds \int_{-\infty}^{+\infty} I_0 \left(k \sqrt{z^2 - (s - r)^2} \right) e^{-|k| l + iky}\, dk =$$
$$= \frac{g_0 l}{4r} \int_{-r_0}^{+r_0} \left(\frac{1}{\sqrt{(l + iy)^2 - z^2 + (s - r)^2}} + \frac{1}{\sqrt{(l - iy)^2 - z^2 + (s - r)^2}} \right) s\, ds \tag{1.31}$$

- Integral (1.31) can be calculated analytically but a final result is very complicated and I do not present it here. But when $r = 0$ or $z = 0$ result simplifies.

1.8. Example (continued)

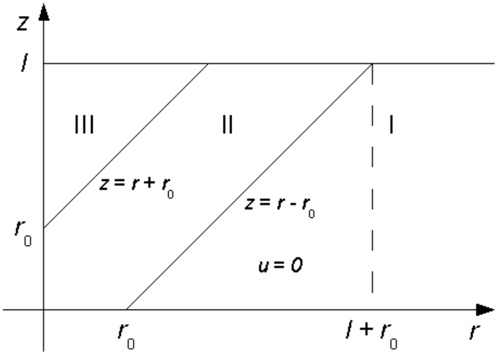

Figure: Phase plane (r, z)

- Rectangle where the solution exists is divided into three domains corresponding to the three stages of process.
- We consider the solution in domain III where it lost hyperbolic properties.

1.8. Example (continued)

We can see that this function is practically independent of its variables. Only in neighbourhood of the boundary $z \approx Z$ the solution sharply increases.

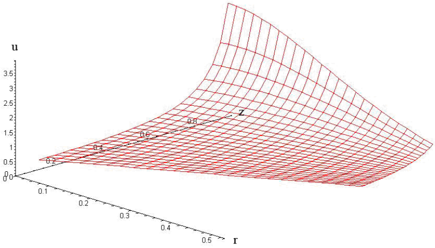

Figure: Plot of the $u(r, 0, z)$ function during the third stage of process

$$u(r,0,z) = \frac{g_0 l}{2}\left\{\ln\frac{\left(\sqrt{(l^2 - z^2) + (r_0^2 - r)^2} + r_0 - r\right)\left(\sqrt{(l^2 - z^2) + (r_0^2 + r)^2} + r_0 + r\right)}{l^2 - z^2} - \right.$$

$$\left. - \frac{4 r_0}{\sqrt{(l^2 - z^2) + (r_0^2 - r)^2} + \sqrt{(l^2 - z^2) + (r_0^2 + r)^2}}\right\}$$

1.8. Example (continued)

$$u(0,y,z)=\frac{g_0 l r_0^3 \cos\left(\frac{3}{2}\arcsin\frac{2ly}{\sqrt{(l^2-y^2-z^2)^2+4l^2y^2}}\right)}{3\left(\left(l^2-(y^2+z^2)\right)^2+4l^2y^2\right)^{\frac{3}{4}}}$$

We see the same behaviour: the solution is practically independent of its variables. Only in neighbourhood of the boundary $z \approx Z$ the solution sharply increases.

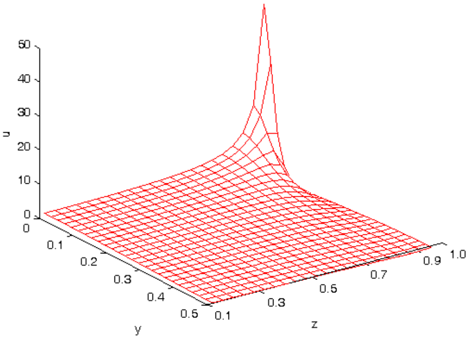

Figure: Plot of the $u(0,y,z)$ function during the third stage of process

2. Problems incorporating requirement of limitation of the solution

2.1. Problems related to the condition of limitation of the solution

- Let us include in formulation of the problem for equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \tag{2.1}$$

  the condition of solution limitation:

$$|u(\mathbf{x}, y, z)| \leqslant m \text{ at } z \geqslant 0. \tag{2.2}$$

- In this case the problem with two conditions at $z = 0$:

$$u(\mathbf{x}, y, 0) = f(\mathbf{x}, y), \tag{2.3}$$

$$\frac{\partial u}{\partial z}(\mathbf{x}, y, 0) = g(\mathbf{x}, y) \tag{2.4}$$

  appears to be over-conditioned and besides it has no solution.

- Here we will try to remove the contradiction in the formulation of the problem at the expense of correction, "modification" of initial condition (2.4).
- The problem (2.1)–(2.3) with one condition (2.3) at $z = 0$ appears to be under-conditioned: no uniqueness is provided.
- The directly opposite problem arises for searching of additional requirements, which allows to take off the uncertainty.

2.1. Problems related to the condition of limitation of the solution (continued)

- Make Fourier transform over variables $\mathbf{x}$ and y:

$$U(\mathbf{p}, k, z) = \frac{1}{(2\pi)^4} \iiiint_{-\infty}^{+\infty} e^{-i(\mathbf{px}+ky)} u(\mathbf{x}, y, z)\, d\mathbf{x}\, dy \tag{2.5}$$

- In the result for Fourier image we find the equation:

$$\frac{d^2U}{dz^2} + (p^2 - k^2)U = 0, \quad p = \sqrt{p_1^2 + p_2^2 + p_3^2}. \tag{2.6}$$

- According to (2.2), it should be supplemented with condition of limitation:

$$|U(\mathbf{p}, k, z)| \leqslant M \text{ at } z \geqslant 0. \tag{2.7}$$

- The properties of the solution of equation (2.6) depend on the sign of the difference $(p^2 - k^2)$. Let us introduce in the four-dimensional space R_4 of variables $(\mathbf{p}, k)$ two sub-areas D_1 and D_2, which are defined by two inequalities

$$D_1 : |k| > p; \quad D_2 : |k| < p. \tag{2.8}$$

- We name the first of them the area of *ellipticity*, the second — the area of *hyperbolicity*.
- In the ellipticity area the bounded solution of equation (2.6) is unique, in respect to the factor. In the area of hyperbolicity its general solution is bounded.
- Existence of such two areas reveals ultrahyperbolic equations dualism.

2.2. Modified Cauchy problem

- Initial conditions (2.3)–(2.4) after Fourier transformation take the form:

$$U(\mathbf{p}, k, 0) = F(\mathbf{p}, k), \tag{2.9}$$

$$\frac{dU}{dz}(\mathbf{p}, k, 0) = G(\mathbf{p}, k), \tag{2.10}$$

where $F(\mathbf{p}, k)$ and $G(\mathbf{p}, k)$ are Fourier images of functions $f(\mathbf{x}, y)$ and $g(\mathbf{x}, y)$ appropriately.

- In the area of hyperbolicity D_2 the solution of the problem (2.6), (2.7), (2.9), (2.10) has the form:

$$U(\mathbf{p}, k, z) = U_2(\mathbf{p}, k, z) = F(\mathbf{p}, k)\cos\left(z\sqrt{p^2 - k^2}\right) + G(\mathbf{p}, k)\frac{\sin\left(z\sqrt{p^2 - k^2}\right)}{\sqrt{p^2 - k^2}}, \quad (\mathbf{p}, k) \in D_2 \tag{2.11}$$

- In the area of ellipticity D_1 the demand of limitation of the solution and two initial conditions are incompatible. Drop the second initial condition (2.10) and write out the bounded solution of equation (2.6), satisfying to condition (2.9) only.

$$U(\mathbf{p}, k, z) = U_1(\mathbf{p}, k, z) = F(\mathbf{p}, k)e^{-z\sqrt{k^2 - p^2}}, \quad (\mathbf{p}, k) \in D_1. \tag{2.12}$$

- At that, according to (2.11), (2.12), we get:

$$\frac{dU}{dz}(\mathbf{p}, k, 0) = \tilde{G}(\mathbf{p}, k) = \begin{cases} -F(\mathbf{p}, k)\sqrt{k^2 - p^2}, & (\mathbf{p}, k) \in D_1, \\ G(\mathbf{p}, k), & (\mathbf{p}, k) \in D_2. \end{cases} \tag{2.13}$$

2.2. Modified Cauchy problem (continued)

- The origin, appropriate to such function U, satisfies equation (2.1), the condition of limitation (2.2), initial condition (2.3), but condition (2.4) is changed to condition:

$$\frac{\partial u}{\partial z}(\mathbf{x}, y, 0) = \tilde{g}(\mathbf{x}, y), \tag{2.14}$$

where $\tilde{g}(\mathbf{x}, y)$ is the original, appropriate to Fourier image $\tilde{G}(\mathbf{p}, y)$ from formula (2.13):

$$\begin{aligned} \tilde{g}(\mathbf{x}, y) = &- \iiiint_{D_1} e^{i(\mathbf{px}+ky)} F(\mathbf{p}, k) \sqrt{k^2 - p^2}\, d\mathbf{p}\, dk + \\ &+ \iiiint_{D_2} e^{i(\mathbf{px}+ky)} G(\mathbf{p}, k)\, d\mathbf{p}\, dk. \end{aligned} \tag{2.15}$$

- The problem for ultrahyperbolic equation (2.1), in which the contradiction between the demand of limitation and two initial conditions is taken off due to the change of condition (2.4) by "corrected" condition (2.14), (2.15), we will call modified Cauchy problem. In such a problem the function $\tilde{g}(\mathbf{x}, y)$ (2.15) is unambiguously determined by initial functions $f(\mathbf{x}, y)$, $g(\mathbf{x}, y)$ and carries inside that part of information about function $g(\mathbf{x}, y)$, which does not contradict to the requirement of limitation.

2.3. Boundary problem

- Let us consider now the problem (2.1)–(2.3) with one condition at $z = 0$. We name it as boundary. After Fourier transform it is reduced to the problem:

$$\frac{d^2U}{dz^2} + (p^2 - k^2)U = 0, \quad p = \sqrt{p_1^2 + p_2^2 + p_3^2}, \tag{2.16}$$

$$|U(p, k, z)| \leqslant M \text{ at } z \geqslant 0, \tag{2.17}$$

$$U(\mathbf{p}, k, 0) = F(\mathbf{p}, k). \tag{2.18}$$

- In the area of ellipticity D_1 the solution of this problem is defined uniquely and is given by formula (2.12):

$$U_1(\mathbf{p}, k, z) = F(\mathbf{p}, k)e^{-z\sqrt{k^2-p^2}}, \quad (\mathbf{p}, k) \in D_1. \tag{2.19}$$

- In inverse Fourier transformation it gives its contribution to the solution through function $u_1(\mathbf{x}, y, z)$:

$$\begin{aligned} u_1(\mathbf{x}, y, z) &= \iiiint_{D_1} e^{i(\mathbf{px}+ky)} U_1(\mathbf{p}, k, z)\, d\mathbf{p}\, dk = \\ &= \iiint_{-\infty}^{+\infty} e^{i\mathbf{px}}\, d\mathbf{p} \int_p^{+\infty} \left\{ e^{+iky} F(\mathbf{p}, k) + e^{-iky} F(\mathbf{p}, -k) \right\} e^{-z\sqrt{k^2-p^2}}\, dk. \end{aligned} \tag{2.20}$$

2.3. Boundary problem (continued)

- Uncertainty of the problem is related to the area of hyperbolicty D_2. The general solution of equation (2.6) in it is bounded and one condition (2.9) for extraction of the unique solution is not enough. It is necessary to include in the formulation of the boundary problem an additional rule for selection of the solution in the area of hyperbolicity D_2.
- Consider two variants as examples.
- *1. Variant with selection of variable y.* Let us take the function

$$U_2(\mathbf{p}, k, z) = F(\mathbf{p}, k)e^{iz\sqrt{p^2 - k^2}\,\mathrm{sgn}(k)}, \tag{2.21}$$

as the solution of equation (2.6) in area D_2 where

$$\mathrm{sgn}(\chi) = \begin{cases} +1, & \chi > 0, \\ -1, & \chi < 0. \end{cases}$$

- After inverse Fourier transformation it gives the following contribution to the answer from the area of hyperbolicity:

$$u_2(\mathbf{x}, y, z) = \iiint_{-\infty}^{+\infty} e^{i\mathbf{px}}\, d\mathbf{p} \int_0^p \left\{ e^{+i\left(ky + z\sqrt{p^2-k^2}\right)} F(\mathbf{p}, k) + e^{-i\left(ky + z\sqrt{p^2-k^2}\right)} F(\mathbf{p}, -k) \right\} dk. \tag{2.22}$$

- Here variables y and z come into powers of exponents as linear combination with positive coefficients:

$$\psi = ky + z\sqrt{p^2 - k^2}. \tag{2.23}$$

- Proportions between coefficients in the process of integration over variables $\mathbf{p}$ and k change, however the tendency of the function u_2 dependence on variables y and z appears to be similar.

2.3. Boundary problem (continued)

- *2. Neutral variant.* Let us take the function

$$U_2(\mathbf{p}, k, z) = F(\mathbf{p}, k)\cos\left(z\sqrt{p^2 - k^2}\right) \tag{2.24}$$

as the solution of equation (2.6) in the area D_2. In this case the contribution to the answer from the area of hyperbolicity has the form:

$$u_2(\mathbf{x}, y, z) = \iiint_{-\infty}^{+\infty} e^{i\mathbf{px}}\, d\mathbf{p} \int_{-p}^{+p} F(\mathbf{p}, k) e^{iky} \cos\left(z\sqrt{p^2 - k^2}\right)\, dk. \tag{2.25}$$

- This variant is equivalent to the modified Cauchy problem with the given initial deviation $f(\mathbf{x}, y)$ and zero initial speed $g(\mathbf{x}, y) = 0$. In respect to variable z the integral (2.25) is the superposition of standing waves.
- Let us summarize the discussion. Any additional requirement, which allows to solve the problem (2.6), (2.7), (2.9) uniquely in the area D_2, takes away the uncertainty in the formulation of the problem (2.1)–(2.3) with one condition at $z = 0$.

2.4. Example

- Let us consider particularities of the solution of the problems under consideration by the example of the second variant of the boundary problem. Let the boundary function f from formula (2.3) be spherically symmetric in the space X_3:

$$f(r,y) = \frac{f_0 l^6}{(l^2+r^2)^2(l^2+y^2)}. \tag{2.26}$$

- Its four-dimensional Fourier image will possess spherical symmetry over variables $\mathbf{p}$:

$$F(\mathbf{p},k) = \frac{f_0 l^4}{16\pi} e^{-(p+|k|)l}. \tag{2.27}$$

- Let us write out solutions of equation (2.6) in the areas of ellipticity D_1 and hyperbolicity D_2:

$$U_1(\mathbf{p},k,z) = F(\mathbf{p},k) e^{-z\sqrt{k^2-p^2}}, \quad (\mathbf{p},k) \in D_1, \tag{2.28}$$

$$U_2(\mathbf{p},k,z) = F(\mathbf{p},k) \cos\left(z\sqrt{p^2-k^2}\right), \quad (\mathbf{p},k) \in D_2, \tag{2.29}$$

- Now let us perform inverse Fourier transform, integrating function U_1 over the area D_1, and function U_2 over the area D_2. Taking into account the spherical symmetry of these functions over p and evenness over k, we get:

$$u(r,y,z) = u_1(r,y,z) + u_2(r,y,z), \tag{2.30}$$

$$u_1(r,y,z) = \frac{f_0 l^4}{2} \int_0^\infty e^{-pl} \psi_0(pr) p^2 \, dp \int_p^\infty e^{-\left(kl + z\sqrt{k^2-p^2}\right)} \cos ky \, dk, \tag{2.31}$$

$$u_2(r,y,z) = \frac{f_0 l^4}{2} \int_0^\infty e^{-pl} \psi_0(pr) p^2 \, dp \int_0^p e^{-kl} \cos\left(z\sqrt{p^2-k^2}\right) \cos ky \, dk, \tag{2.32}$$

$$\psi_0 = \sqrt{\frac{\pi}{2x}} J_{1/2}(x) = \frac{\sin x}{x}.$$

2.4. Example (continued)

- Integrals (2.31) and (2.32) can be reduced to one-dimensional integrals and simplified but calculated to the end only at $r = 0$, $y = 0$:

$$u_1(0,0,z) = f_0 \left\{ \frac{l^2(z^4 + z^3 l + 22z^2 l^2 - 30zl^3 - 60l^4)}{8z^5(z+l)} - \frac{3l^4(z^2 - 5l^2)}{2z^6} \ln\left(1 + \frac{z}{l}\right) \right\}, \tag{2.33}$$

$$u_2(0,0,z) = f_0 \left\{ -\frac{l^2(z^8 + 25z^6 l^2 - 47z^4 l^4 - 69z^2 l^6 - 30l^8)}{8z^4(z^2+l^2)^3} + \frac{3l^4(z^2 - 5l^2)}{4z^6} \ln\left(1 + \frac{z^2}{l^2}\right) \right\}. \tag{2.34}$$

- Let us expand functions (2.27)–(2.28) and their sum over inverse powers of z to explore their asymptotics in the limit of $z \to \infty$:

$$u_1(0,0,z) \approx \frac{f_0 l^2}{8z^2}, \quad u_2(0,0,z) \approx -\frac{f_0 l^2}{8z^2}, \tag{2.35}$$

$$u_1(0,0,z) + u_2(0,0,z) \approx -f_0 \frac{8l^5}{z^5}. \tag{2.36}$$

- Here the interesting phenomenon has occurred. Functions u_1 and u_2 behave themselves at $z \to \infty$ as z^{-2}, whoever in their sum three first terms cancel. In the result its sum reduces at infinity as z^{-5}.

2.4. Example (continued)

$$u_1(0,0,z) = f_0\left\{\frac{l^2(z^4+z^3l+22z^2l^2-30zl^3-60l^4)}{8z^5(z+l)} - \right.$$
$$\left. - \frac{3l^4(z^2-5l^2)}{2z^6}\ln\left(1+\frac{z}{l}\right)\right\},$$

$$u_2(0,0,z) = f_0\left\{-\frac{l^2(z^8+25z^6l^2-47z^4l^4-69z^2l^6-30l^8)}{8z^4(z^2+l^2)^3} + \right.$$
$$\left. + \frac{3l^4(z^2-5l^2)}{4z^6}\ln\left(1+\frac{z^2}{l^2}\right)\right\}.$$

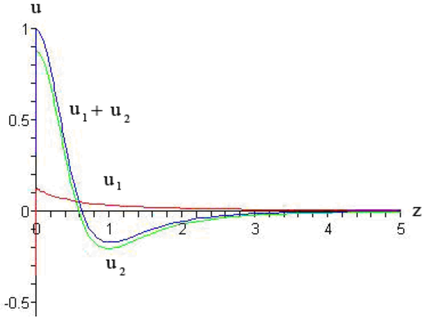

Figure: Plot of the functions $u_1(0,0,z)$ from (2.33) and $u_2(0,0,z)$ from (2.34) and their sum